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A. H. M. LEVELT

A descent theorem for differential operators of the first kind

To N. G. de Bruijn

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Katholieke Universiteit, Nijmegen, The Netherlands

1. INTRODUCTION

In my thesis ([8]), prepared under N. G. de Bruijn's supervision, I studied the structure of the solution space of a system of ordinary linear differential equations in a neighbourhood of a regular singularity of the first kind. The classical and well-known structure theorem was deduced from an abstract characterization of the solution space in an essentially algebraic way.

In the last decade (singularities of) analytic and algebraic differential equations have aroused the interest of several mathematicians (e.g. [1], [4], [7], [9], [10]) and various algebraic methods have been developed. Usually solutions in a neighbourhood of a singularity are studied by applying existence and uniqueness theorems in the near-by regular points. For an algebraic mind it is not too far-fetched to conceive the theorems in question in terms of "descent". The purpose of the present paper is to show that this idea works also at the singular point(s). We shall state as a first result in this direction a simple and useful theorem and we shall show how this theorem leads to known and new results in a straight-forward way. In the "classical" situation differential equations with a singularity of the first kind are studied with respect to the field of complex numbers. A natural question is that of dependence on parameters. This amounts to replacing $\mathbb{C}$ by some other field or ring. Our theorem expresses the somewhat surprising fact that the classical results still hold, when

$\mathbb{C}$ is replaced by any local ring such that the residue field has characteristic 0. The theorem is a "one variable" version which enables us to treat the case of more variables by induction on the number of variables in more or less the same way as Cauchy's theorem can be proved in the absence of singularities.

The theory in this paper is still seriously restricted by the condition "no positive integer differences". Another restriction is that to formal power series. Encouraged by the experience that "everything formal is convergent" in the realm of regular singularities, we may expect that analogous results are valid in the analytic situation.

I was lead to the idea of the theorem by reading A. R. P. van den Essen's work on regular singularities and on Fuchsian modules (cf [5], [6]).

2. STATEMENT OF THE THEOREM

Before stating the main result we introduce some notions and conventions. All rings in this paper are commutative and have a unit element. A derivation η of a ring B is an additive map $\eta: B \to B$ such that $\eta(b_1b_2) = b_1\eta(b_2) + b_2\eta(b_1)$, all $b_1, b_2 \in B$.

DEFINITION 1. Let B be a ring, η a derivation of B and E a B-module. A map $S: E \to E$ is called a differential operator on E with respect to η, when S is additive and satisfies

$$S(bx) = \eta(b)x + bSx$$

all $b \in B$, $x \in E$.

REMARK. When η is the trivial derivation $\eta(b) = 0$, all $b \in B$, then S is a B-linear endomorphism.

From now on A will be a ring and $B = A[[t]]$ the ring of formal power series in t with coefficients in A. By θ we denote the derivation $t\dfrac{d}{dt}$ on B which annihilates the elements of A.

DEFINITION 2. Let E be a B-module. A differential operator of the first kind on E is a differential operator with respect to θ.

It follows from this definition that $D(t^iE) \subset t^iE$ for any $i \in \overline{\mathfrak{N}} = \mathfrak{N} \cup \{0\}$ So there is an induced A-linear map $\tilde{D}: \tilde{E} = E/tE \to \tilde{E}$. In the remainder of this section we assume that A is a local ring with maximal ideal $\mathfrak{m}$ and that the residue field $k = A/\mathfrak{m}$ has characteristic 0. B is also a local ring and the maximal ideal $\mathfrak{n}$ consists of the formal power series with constant coefficient in $\mathfrak{m}$. When E is a B-module of finite type, then $\tilde{E}$ obviously is an A-module of finite type and $\bar{E} = k \otimes_A \tilde{E} = \tilde{E}/\mathfrak{m}\tilde{E} = E/\mathfrak{n}E$ is a k-vector space of finite dimension. $\tilde{D}$ induces a k-linear endomorphism $\bar{D}$ of $\bar{E}$.

DEFINITION 3. Let E be a B-module of finite type and D a differential operator of the first kind on E. Then D is said to have the property $NPID$ (or to satisfy the condition $NPID$), if no two eigenvalues (in an algebraic closure of k) of $\bar{D}$ differ by a positive integer.

THEOREM. Let E be a B-module of finite type such that t is no zero-divisor in E and $\bigcap_{i=0}^{\infty} t^i E = 0$. Let D be a differential operator of the first kind on E having the property $NPID$. Then the following assertions hold:

(i) There exists an A-submodule E_0 of E such that $D(E_0) \subset E_0$ and $E = E_0 + tE$, a direct sum of A-submodules.

(ii) When F_0 is an A-submodule of E such that $D(F_0) \subset F_0$ and $F_0 \cap tE = 0$, then $F_0 \subset E_0$.

(iii) Let η be a derivation of B such that $\eta(A) \subset A$, and S a differential operator on E with respect to η which commutes with D. Then $S(E_0) \subset E_0$ and S is uniquely determined by η and its restriction to E_0.

REMARK. It follows from (ii) that E_0 in (i) is unique. Taking $S = D$ in (iii) we see that D is completely determined by its restriction to E_0. Since E_0 and $\bar{E}$ are isomorphic A-modules, E_0 is an A-module of finite type. The condition $\bigcap_{i=0}^{\infty} t^i E = 0$ is automatically satisfied e.g. when A is Noetherian or when E is a free B-module.

3. PROOF OF THE THEOREM

In this section A will be a local ring, $\mathfrak{m}$ its maximal ideal and k its residue field, which will be of characteristic 0.

LEMMA 1. Let M be an A-module of finite type, $\delta \in \mathrm{End}_A(M)$ $f, g \in A[X]$ monic polynomials such that $\bar{f}, \bar{g} \in k[X]$ are relatively prime. (By $\bar{f}$ we denote the polynomial obtained by applying the residue map to the coefficients of f). Assume $f(\delta) = 0$. Then $g(\delta)$ is an automorphism of M.

PROOF. In virtue of [3] Chap. III, § 4, Prop. 2 there exist polynomials $a, b \in A[X]$ such that $1 = af + bg$. Substituting $X = \delta$ we find $1_M = b(\delta)g(\delta)$, since $f(\delta) = 0$.

LEMMA 2. Let M be an A-module of finite type, $\delta \in \mathrm{End}_A(M)$ $\bar{\delta} \in \mathrm{End}_k(\bar{M})$ the map induced by δ. Then there exists a monic polynomial $f \in A[X]$ such that $f(\delta) = 0$ and $\bar{f}$ is the characteristic polynomial of $\bar{\delta}$.

PROOF. There exists a surjective A-homomorphism $u: L \to M$, where L is free of finite rank, and such that $\bar{u}: \bar{L} \to \bar{M}$ is an isomorphism. Lift $\bar{\delta}$ to an A-endomorphism ε of L, and let f be the characteristic polynomial of ε. Then it follows that $f(\delta) = 0$. Moreover, $\bar{f} \in k[X]$ is the characteristic polynomial of $\bar{\varepsilon} \in \mathrm{End}_k(\bar{L})$ and $\bar{\varepsilon}$ can be identified with $\bar{\delta}$, since $\bar{u}$ is an isomorphism.

PROOF OF THE THEOREM. For any $g \in A[X]$ $g(D)$ is an A-linear endomorphism of E which maps t^iE into t^iE, $i \in \overline{\mathbf{N}}$, and commutes with any A-linear operator which commutes with D. In virtue of Lemma 2 there exists a monic polynomial $f \in A[X]$ such that $f(\tilde{D})=0$ and $\bar{f} \in k[X]$ is the characteristic polynomial of $\bar{D}$. Define $\Phi = f(D)$. Φ induces $\tilde{\Phi}=f(\tilde{D})$: $\tilde{E} \to \tilde{E}$, and this map is the zero map, whence $\Phi(E) \subset tE$. We shall now prove that $\Phi : tE \to tE$ is a bijection. First of all for $i \in \overline{\mathbf{N}}$ the map $m: E \to t^iE$ defined by $m(x)=t^ix$ is an isomorphism, t being no zero-divisor in E. m induces the isomorphism $\tilde{m}: \tilde{E} \to \tilde{E}_i = t^iE/t^{i+1}E$. A trivial computation shows that the diagram

$$\begin{tikzcd} E \arrow[r, "m"] \arrow[d, "D+i"'] & t^iE \arrow[d, "D"] \\ E \arrow[r, "m"] & t^iE \end{tikzcd}$$

is commutative. ($D+i$ means $D+i.1_E$). This leads to another commutative diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{m}"] \arrow[d, "\tilde{D}+i"'] & \tilde{E}_i \arrow[d, "\tilde{D}_i"] \\ \tilde{E} \arrow[r, "\tilde{m}"] & \tilde{E}_i \end{tikzcd}$$

where $\tilde{D}_i$ denotes the map induced by the restriction D_i of D to t^iE. From this diagram we finally deduce a last one "by applying f"

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{m}"] \arrow[d, "f(\tilde{D}+i)"'] & \tilde{E}_i \arrow[d, "\tilde{\Phi}_i"] \\ \tilde{E} \arrow[r, "\tilde{m}"] & \tilde{E}_i \end{tikzcd}$$

$\tilde{\Phi}_i = f(\tilde{D}_i)$ can also be described as the map induced by the restriction Φ_i of Φ to t^iE. From now on we suppose $i \geqslant 1$. We want to prove that $\tilde{\Phi}_i$ is a bijection. For this it is sufficient that $f(\tilde{D}+i)$ is a bijection. Define $g=f(X+i) \in A[X]$. Since D has the property $NPID$ $\bar{f}$ and $\bar{g}$ are relatively prime. Now apply Lemma 1 with $M=\tilde{E}$, $\delta=\tilde{D}$. It follows that $f(\tilde{D}+i)$ is an automorphism of $\tilde{E}$. So we have proved that the map $gr(\Phi)$ induced by Φ in $gr(tE) = \bigoplus_{i=1}^{\infty} \tilde{E}_i$ is a bijection, and because tE is complete and separated in the t-adic topology $\Phi: tE \to tE$ is bijective (cf [3], Chap. III, § 2, Théorème 1, Corollaire 3). Now we can prove the assertions:

(i) Define $E_0 = \operatorname{Ker} \Phi$. Obviously, E_0 is an A-submodule of E, and $D(E_0) \subset E_0$ because Φ and D commute. When $x \in E$, $\Phi(x)=y \in tE$ and because $\Phi : tE \to tE$ is a bijection there exists $z \in tE$ such that $y=\Phi(z)$.

It follows that $\Phi(x-z)=\Phi(x)-\Phi(z)=y-y=0$, whence $x-z\in E_0$. Consequently $E=E_0+tE$. Now let x be an element of $E_0\cap tE$. Then $\Phi(x)=0$ because $x\in E_0$. However, 0 is the only element of tE which is mapped onto 0 by Φ, whence $x=0$. This finishes the proof of (i).

(ii) Let x be an element of F_0. Because $D(F_0)\subset F_0$ $g(D)(F_0)\subset F_0$ for any $g\in A[X]$. So we have $\Phi(x)\in F_0$. Now Φ maps E into tE and $F_0\cap tE=0$, which shows $\Phi(x)=0$, i.e. $x\in E_0$.

(iii) If $f=X^n+a_1X^{n-1}+\ldots+a_n$, a simple computation yields

$$S\Phi=\Phi S+\sum_{i=0}^{n}\eta(a_i)D^{n-i}.$$

We shall prove $Sx\in E_0$, when $x\in E_0$. Applying the above relation we see

$$0=\Phi Sx+\sum_{i=0}^{n}\eta(a_i)D^{n-i}x,$$

and the second term in the last member represents an element of E_0. (Here we have used $\eta(A)\subset A$.) On the other hand $\operatorname{Im}\Phi\subset tE$, and since $E_0\cap tE=0$, we conclude that $\Phi Sx=0$. Hence $Sx\in E_0$ as we wanted to prove. Let S' be a second differential operator on E with respect to η and suppose that S' also satisfies the hypotheses of (iii), and that on E_0 S and S' coincide. We shall prove that $S=S'$. Define $T=S-S'$. Then T is a B-linear endomorphism of E which vanishes on E_0. Hence T vanishes on BE_0, and as we shall show $BE_0=E$. This will complete the proof of (iii) and of the Theorem. Notice that BE_0 is a B-module of finite type and that $E\subset BE_0+tE$. In virtue of Nakayama's lemma it follows that $E=BE_0$.

4. APPLICATIONS, GENERALIZATIONS, DISCUSSION

A. In the situation of the Theorem we want to describe E as a tensor product. Let E^* be the B-module $E^*=B\otimes_A E_0$, E^* is a B-module of finite type, and we can define operators D^*, S^* on E^* by the formulas

$$D^*(b\otimes x)=(\theta b)\otimes x+b\otimes Dx,$$
$$S^*(b\otimes x)=(\eta b)\otimes x+b\otimes Sx.$$

Obviously, D^* is well-defined, and the reader should have no difficulty in showing that D^* is in fact a differential operator of the first kind. That S^* is well-defined follows from

$$\begin{aligned}\eta(ab)\otimes x+(ab)\otimes Sx&=(a\eta b)\otimes x+(b\eta a)\otimes x+(ab)\otimes Sx\\&=(\eta b)\otimes ax+b\otimes(\eta a)x+b\otimes aSx=(\eta b)\otimes ax+b\otimes S(ax),\end{aligned}$$

when $a\in A$, $b\in B$, $x\in E_0$. S^* is a differential operator on E^* with respect to η. Notice that D^* and S^* are determined by the action of D and S, respectively, on E_0. The multiplication map $\mu: E^*\to E$ is defined by $\mu(b\otimes x)=bx$. This is a homomorphism of B-modules and the image is

$BE_0 = E$. So μ is surjective. Let K be the kernel of μ. If $x \in K$, there exist $b_1, \ldots, b_n \in B$, $x_1, \ldots, x_n \in E_0$ such that $x = \sum_{i=1}^n b_i \otimes x_i$, and $\mu(x) = 0$ implies $\sum_{i=1}^n b_i x_i = 0$. Now write $b_i = a_i + tc_i$, where $a_i \in A$, $c_i \in B$. Then $0 = \sum_{i=1}^n b_i x_i = \sum_{i=1}^n a_i x_i + t \sum_{i=1}^n c_i x_i$, a sum of a term in E_0 and one in tE. Since $E_0 \cap tE = 0$ we conclude $\sum a_i x_i = 0$ and $\sum c_i x_i = 0$, t not being a zero-divisor in E. Define $y \in E^*$ by $y = \sum c_i \otimes x_i$. Obviously, $y \in K$ and

$$\begin{aligned} x &= \sum b_i \otimes x_i = \sum a_i \otimes x_i + \sum tc_i \otimes x_i \\ &= \sum 1 \otimes a_i x_i + t \sum c_i \otimes x_i = ty. \end{aligned}$$

It follows that $K = tK$, and by iteration $K = \bigcap_{i=0}^{\infty} t^i K \subset \bigcap_{i=0}^{\infty} t^i E^*$. On the other hand, in virtue of the condition $\bigcap_{i=0}^{\infty} t^i E = 0$, every element of $\bigcap_{i=0}^{\infty} t^i E^*$ is in the kernel of μ. So we have proved $\operatorname{Ker} \mu = \bigcap_{i=0}^{\infty} t^i E^*$. One can invent several conditions implying the vanishing of this intersection, e.g. A Noetherian or E free B-module. Notice that D, D^* (S, S^*, respectively) are compatible with μ, i.e.

$$\mu \circ D^* = D \circ \mu, \quad \mu \circ S^* = S \circ \mu.$$

Finally, we mention the fact that E can be identified with the B-module of "formal power series in t with coefficients in E_0". When $x = \sum_{i=0}^{\infty} t^i x_i$, $x_i \in E_0$, is an element of E, then

$$Dx = \sum_{i=0}^{\infty} t^i (D+i) x_i,$$

$$Sx = \sum_{i=0}^{\infty} (it^{i-1} \eta(t) x_i + t^i S x_i).$$

B. With a view to further applications we make a remark concerning assertion (iii) of the Theorem.

The condition there that D and S commute has a strong implication for θ and η: an easy computation shows that for any $b \in B$, $x \in E$ the following relation holds

$$\theta\eta(b)x = \eta\theta(b)x$$

even without assuming that $\eta(A) \subset A$. When no element $\neq 0$ of B annihilates E, it follows that θ and η commute. The converse is not true: if θ and η commute, it doesnot follow that D and S commute. However, the commutation of θ and η implies $\eta(A) \subset A$. For writing

$$\eta(a) = a_0 + a_1 t + a_2 t^2 + \ldots, \quad a, a_i \in A,$$

we have

$$0 = \eta(0) = \eta(\theta a) = \theta\eta(a) = a_1 t + 2a_2 t^2 + \ldots$$

whence $a_1 = a_2 = \ldots = 0$.

C. We shall generalize the Theorem to the case of more variables. So let R be a local ring with residue field k of characteristic 0, r a positive integer, $B = R[[t_1, \ldots, t_r]]$ the ring of formal power series in $t_1, \ldots, t_r$ with coefficients in R and E a B-module of finite type such that $t_1 \ldots t_r$ is no zero-divisor E. Moreover, we suppose that $\bigcap_{i=0}^{\infty} \mathfrak{t}^i E = 0$, $\mathfrak{t}$ denoting the ideal of B generated by $t_1, \ldots, t_r$. On E we consider a set of commuting differential operators of the first kind $D_1, \ldots, D_r$, i.e.

$$D_i(bx) = (\theta_i b)x + bD_i x$$

all $i \in \{1, \ldots, r\}$, $b \in B$, $x \in E$ (θ_i denotes the derivation $t_i \dfrac{\partial}{\partial t_i}$ on B). Denoting the maximal ideal of B by $\mathfrak{n}$, we see that D_i maps $\mathfrak{n}E$ into itself, whence an induced k-linear map $\bar{D}_i$ of the finite demensional k-vector space $\bar{E} = E/\mathfrak{n}E$. We shall say that the condition $NPID$ is satisfied for $D_1, \ldots, D_r$ when no $\bar{D}_i$ has eigenvalues differing by a positive integer. With these definitions and hypotheses we can state:

(i′) There exists an R-submodule E_0 of E such that $E = E_0 + \mathfrak{t}E$ is a direct sum of R-modules and $D_i(E_0) \subset E_0$, all $i \in \{1, \ldots, r\}$.

(ii′) If F_0 is an R-submodule of E such that $F_0 \cap \mathfrak{t}E = 0$ and $D_i(F_0) \subset F_0$ all i, then $F_0 \subset E_0$.

(iii′) Let S be a differential operator on E with respect to a derivation η of B which commutes with all θ_i. Suppose that S commutes with all D_i. Then $S(E_0) \subset E_0$, and S is uniquely determined by η and its restriction to E_0.

Observe that the case $r = 1$ is just our Theorem. The assertions can easily be proved by induction on r. Suppose the assertions are valid for less than r variables. Writing $R' = R[[t_r]]$, we can apply the induction hypothesis to the situation where r is replaced by $r-1$, by R', etc., the condition $NPID$ being satisfied for $D_1, \ldots, D_{r-1}$. So there exists a R'-submodule E' of E such that $E = E' + (t_1, \ldots, t_{r-1})E$ is a direct sum of R'-modules, and E' is invariant under $D_1, \ldots, D_{r-1}$. When S satisfies the conditions of (iii′), then $S(E') \subset E'$. Notice that D_r satisfies these conditions, whence $D_r(E') \subset E'$. Define $F' = R'F_0$. F' is an R'-submodule of E and $D_i(F') \subset F'$, all i. Put $G = F' \cap (t_1, \ldots, t_{r-1})E$. G is an R'-submodule of E. We claim that $G = 0$. Suppose $f = \sum_{i=1}^{s} a_i f_i \in G$, where $f_i \in F_0$ and $a_i \in R'$. Writing $a_i = a_{i0} + t_r b_i$, where $a_{i0} \in R$, $b_i \in R'$ and $f' = \sum_{i=1}^{s} a_{i0} f_i$, $g = \sum_{i=1}^{s} b_i f_i$ we have $f = f' + t_r g$. Notice that $f' \in F_0$ and $g \in F'$. Since $f' = f - t_r g \in$ $\in (t_1, \ldots, t_r)E$ and $F_0 \cap (t_1, \ldots, t_r)E = 0$, we see that $f' = 0$, whence $f = t_r g$. Using the fact that $E = E' + (t_1, \ldots, t_{r-1})E$ is a direct sum of R'-modules we deduce that $g \in G$. So we have proved $G \subset t_r G$. This yields $G \subset t_r^i G \subset \mathfrak{t}^i E$, all i. In virtue of $\bigcap_{i=0}^{\infty} \mathfrak{t}^i E = 0$ we finally have $G = 0$, whence $F_0 \subset F' \subset E'$ by the induction hypothesis. Now we can apply the Theorem with A replaced by R, t by t_r, B by $R[[t_r]]$, E by E', D by D_r, whcreas the restrictions of $S, D_1, \ldots, D_{r-1}$ to E' satisfy the hypothesis of (iii) of the Theorem. Consequently, there exists an R-submodule E_0 of E', such that

$E' = E_0 + t_r E'$, a direct sum of R-modules, and E_0 is invariant under all D_i and S. Since $F_0 \cap t_r E' = 0$, we also have $F_0 \subset E_0$.

Putting together the preceding results we see that

$$E = E_0 + t_r E' + (t_1, \ldots, t_{r-1})E$$

is a direct sum of R-modules. Obviously,

$$t_r E' + (t_1, \ldots, t_{r-1})E \subset (t_1, \ldots, t_r)E,$$

and the opposite inclusion follows from

$$\begin{aligned}(t_1, \ldots, t_r)E &= (t_1, \ldots, t_{r-1})E + t_r E = \\ &= (t_1, \ldots, t_{r-1})E + t_r(E' + (t_1, \ldots, t_{r-1})E) \subset t_r E' + (t_1, \ldots, t_{r-1})E.\end{aligned}$$

This proves (i'). Finally, by induction S is uniquely determined by $S|_{E'}$, and again this latter operator is uniquely determined by its restriction to E_0. This completes the proof.

Special cases of the result obtained here are known (cf [7], Théorème 3.4 and also [5], Theorem 4.2 which can be deduced quite easily from the above results). The fact that E turns out to be a free B-module in some cases is a trivial consequence of the isomorphism $B \otimes_R E_0 \to E$, when R is a field (or a discrete valuation ring).

Replacing in the foregoing $D_1, \ldots, D_r$ by a set of commuting differential operators $P_1, \ldots, P_r$ with respect to $\dfrac{\partial}{\partial t_1}, \ldots, \dfrac{\partial}{\partial t_r}$ we find a generalization of Cauchy's theorem as mentioned in the introduction. We omit the details. For the proof put $D_i = t_i P_i$, all i. The condition $NPID$ is automatically satisfied.

D. What can be said in the case of non-local A? We shall give a partial answer by analyzing those parts of the proof of the Theorem where the local character of A plays a role. The situation to study is the following: A any commutative ring, $B = A[[t]]$, E a B-module of finite type, such that t is no zero-divisor in E and $\bigcap_{i=0}^{\infty} t^i E = 0$. Furthermore $D: E \to E$ a differential operator of the first kind. Now some condition of the type $NPID$ should be imposed. In order to formulate such a condition we first observe that there still exists a monic polynomial $f \in A[X]$ such that $f(\tilde{D}) = 0$. Let Ω be the set of maximal ideals of A. For any $\mathfrak{m} \in \Omega$ we denote by $\varkappa(\mathfrak{m})$ the residue field and we suppose that all residue fields have characteristic 0. This will be the case e.g. when A contains a field of characteristic 0. We denote by $f^{(\mathfrak{m})} \in \varkappa(\mathfrak{m})[X]$ the polynomial obtained from f by reducing the coefficients modulo $\mathfrak{m}$. $\tilde{D}$ induces a $\varkappa(\mathfrak{m})$-linear map $D(\mathfrak{m})$ in $E(\mathfrak{m}) = \tilde{E}/\mathfrak{m}\tilde{E}$ and obviously $f^{(\mathfrak{m})}(D(\mathfrak{m})) = 0$, though $f^{(\mathfrak{m})}$ need not be the characteristic polynomial of $D(\mathfrak{m})$. Nevertheless, it seems natural to require that for no $\mathfrak{m} \in \Omega$ the zeroes of $f^{(\mathfrak{m})}$ differ by a positive integer. This will now be our condition $NPID$. The critical point in the proof of the Theorem is the bijectivity of $f(\tilde{D} + i): \tilde{E} \to \tilde{E}$, when i

is a positive integer. It suffices to show that for all $\mathfrak{m} \in \Omega$ the map obtained by localizing

$$f_{\mathfrak{m}}(\tilde{D}_{\mathfrak{m}}+i): \tilde{E}_{\mathfrak{m}} \to \tilde{E}_{\mathfrak{m}}$$

is a bijection (cf [2], Chap. II, § 3, no 3, Théorème 1). Here $\tilde{E}_{\mathfrak{m}}$ denotes the $A_{\mathfrak{m}}$-module obtained by localizing $\tilde{E}$ in $\mathfrak{m}$, $f_{\mathfrak{m}} \in A_{\mathfrak{m}}[X]$ the polynomial obtained by applying the canonical map $A \to A_{\mathfrak{m}}$ to the coefficients of f, and $\tilde{D}_{\mathfrak{m}}: \tilde{E}_{\mathfrak{m}} \to \tilde{E}_{\mathfrak{m}}$ the localization of the A-homomorphism $\tilde{D}$. To this situation we can apply Lemma 1 of section 3. The two polynomials in question are $f_{\mathfrak{m}}(X)$ and $f_{\mathfrak{m}}(X+i)$ and $\delta=\tilde{D}_{\mathfrak{m}}$ and it suffices to notice that the residue field of $A_{\mathfrak{m}}$ is canonical isomorphic with $\varkappa(\mathfrak{m})$, and that the residue map $A_{\mathfrak{m}} \to \varkappa(\mathfrak{m})$ induces a map $A_{\mathfrak{m}}[X] \to \varkappa(\mathfrak{m})[X]$ which takes $f_{\mathfrak{m}}$ to $f^{(\mathfrak{m})}$. The conclusion is that $f_{\mathfrak{m}}(\tilde{D}_{\mathfrak{m}}+i)$ is an automorphism of $\tilde{E}_{\mathfrak{m}}$. Hence $f(\tilde{D}+i)$ is an automorphism of $\tilde{E}$. It is now evident that the assertions (i), (ii) and (iii) of the Theorem hold in the more general situation considered here. There is only one other point where the proof should be adapted viz. the relation $E=BE_0$. Again it suffices to prove that $E_{\mathfrak{m}}=B_{\mathfrak{m}}(E_0)_{\mathfrak{m}}$ and this follows by applying Nakayama's lemma to the relation $E_{\mathfrak{m}} \subset B_{\mathfrak{m}}(E_0)_{\mathfrak{m}}+tE_{\mathfrak{m}}$ which is an immediate consequence of $E=E_0+tE$.

The result obtained looks quite satisfactory. In fact it is not so much. The condition *NPID* is formulated in terms of f which polynomial is not uniquely determined in general. It would be preferable to impose a modified condition: For all $\mathfrak{m} \in \Omega$ the map $D(\mathfrak{m})$ has no eigenvalues differing by a positive integer. This condition is completely intrinsic and there is reason to believe that the assertions of the Theorem still hold under this weaker condition (which probably does not imply the existence of f as required in this section). One might try to obtain this result by applying the local version to the localized situation and "glue together" the $A_{\mathfrak{m}}$-modules $(E_{\mathfrak{m}})_0$ to an A-module E_0. Precisely the uniqueness of the modules $(E_{\mathfrak{m}})_0$ should garantuee the success of this procedure. There are, however, some technical points asking for a careful treatment (viz. $A[[t]])_{\mathfrak{m}} \neq A_{\mathfrak{m}}[[t]])$.

E. An important question is the following: What can be said if the condition *NPID* is not satisfied? The reader might convince himself that the assertions no longer hold by studing the counter-example: $A=k$ is a field of characteristic 0, $B=k[[t]]$, E the free B-module on the basis (e_1, e_2), and $D: E \to E$ defined by $De_1=e_1+te_2$, $De_2=0$. There exist also positive results. When A is a field of characteristic 0, then E contains a B-submodule F with the following properties (i) $D(F) \subset F$ (ii) $D|_F$ has the property *NPID* (iii) $t^qE \subset F$ for some $q \in \overline{\mathfrak{N}}$ (cf. [7], Théorème 3.4, a several variables version). Using Jordan decomposition of D a similar result can be proved under rather special conditions, e.g. A complete, local k-algebra, E free B-module of finite rank. At this moment it is not clear how the technic of the present paper could be used when $NPID$ is dropped. Since the cases A complete, and A = field appear to be simpler, one could try to reduce the case of A general local ring to the cases mentioned earlier. This requires a study of base change $A \to A'$.

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